Symmetry Restoration and Quantum Mpemba Effect in Symmetric Random Circuits

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Entanglement asymmetry, which serves as a diagnostic tool for symmetry breaking and a proxy for thermalization, has recently been proposed and studied in the context of symmetry restoration for quantum many-body systems undergoing a quench. In this Letter, we investigate symmetry restoration in various symmetric random quantum circuits, particularly focusing on the U(1) symmetry case. In contrast to nonsymmetric random circuits where the U(1) symmetry of a small subsystem can always be restored at late times, we reveal that symmetry restoration can fail in U(1)-symmetric circuits for certain weak symmetry-broken initial states in finite-size systems. In the early-time dynamics, we observe an intriguing quantum Mpemba effect implying that symmetry is restored faster when the initial state is more asymmetric. Furthermore, we also investigate the entanglement asymmetry dynamics for SU(2) and Z_2 symmetric circuits and identify the presence and absence of the quantum Mpemba effect for the corresponding symmetries, respectively. A unified understanding of these results is provided through the lens of quantum thermalization with conserved charges.

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Introduction—Quantum thermalization is an important topic of fundamental interest. The thermalized state of a chaotic quantum many-body system is linked to the unitary evolution by the eigenstate thermalization hypothesis (ETH) [1–5]. Specifically, the reduced density matrix of a small subsystem A equilibrates to a canonical ensemble: $\rho_A \propto e^{-\beta \hat{H}_A}$ where $\hat{H}_A$ is the system-of-interest Hamiltonian. Furthermore, if $\hat{H}_A$ respects U(1) symmetry, the subsystem equilibrates to a grand canonical ensemble: $\rho_A \propto e^{-\beta \hat{H}_A - \mu \hat{Q}_A}$ with μ denoting the chemical potential and $\hat{Q}_A$ representing the conserved charge operator. Consequently, $[\hat{Q}_A, \rho_A] = 0$, and the weak symmetry (also known as average symmetry) for subsystem A can always be restored even with a U(1) symmetry-broken initial state. In other words, symmetry restoration for the small subsystem under quench is an indicator of quantum thermalization. This relation also holds for non-Abelian symmetries with noncommuting conserved charges [6].

In addition to the late-time or equilibrium behaviors, nonequilibrium dynamics have attracted significant attention due to rich interesting phenomena. One example is the counterintuitive Mpemba effect [25], which states that hot water freezes faster than cold water and has been extended in various systems [26–33]. Quantum versions of the Mpemba effect have also been extensively investigated [34–42] where an external reservoir driving the system out of equilibrium is necessary for the emergence of Mpemba effects. Recently, an intriguing anomalous relaxation phenomenon has been observed in isolated quantum integrable systems [43]. The U(1) symmetry-broken initial states are evolved with the U(1) symmetric Hamiltonian and the weak U(1) symmetry restoration for subsystem A is observed when the subsystem size $|A|$ is less than half of the total system size N [44–54]. More importantly, symmetry restoration occurs more rapidly for more asymmetric initial states. This phenomenon is dubbed as the quantum Mpemba effect (QME) [43] and has been demonstrated experimentally on quantum simulation platforms [55]. However, a comprehensive investigation of symmetry restoration and QME in generic chaotic systems is lacking.

Previous work has demonstrated that the U(1) symmetry of ρ_A with $|A| < N/2$ can be restored when the whole system is the random Haar state [56], which can be regarded as the output state of random Haar circuit without U(1) symmetry at late times. This raises a natural question regarding the existence of the QME in the dynamics of random Haar circuits with and without the corresponding symmetry. Moreover, previous investigations on QME have primarily focused on the U(1) symmetry restoration in integrable Hamiltonian dynamics [43,55,57,58] where the theoretical explanations of QME have hinged on integrability [58]. Although the nonequilibrium dynamics after a global Z_2 symmetric quantum quench has been investigated before [59], no QME has been observed. Therefore, it remains an open question of whether QME manifests in the dynamics for other alternative symmetry restoration and

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whether symmetric random quantum circuits [60–76] play a similar role as the quenched Hamiltonian dynamics. Last but not least, a unified theoretical understanding of symmetry restoration, the QME, and its relation with thermalization is still elusive.

In this Letter, we investigate the dynamics of subsystem symmetry restoration across a range of symmetric and nonsymmetric quantum random circuits, considering different initial states. To quantify the degree of symmetry breaking in subsystem A, we employ the concept of entanglement asymmetry (EA) [43,54,59,77], which has been extensively studied as an effective symmetry broken measure in out-of-equilibrium many-body systems [58,78] and quantum field theories [79–81]. It is defined as

$$\Delta S_A = S(\rho_{A,Q}) - S(\rho_A). \tag{1}$$

Here, $S(\rho_A)$ represents the von Neumann entropy of subsystem ρ_A, and $\rho_{A,Q} = \sum_q \Pi_q \rho_A \Pi_q$, where Π_q is the projector to the qth eigensector of the corresponding symmetry operator g. In the case of U(1) symmetry, the symmetry operator g is the total charge operator in subsystem A, $\hat{Q}_A = \sum_i^{|A|} \sigma_i^z$, and the computational basis coincides with the eigenbasis of $\hat{Q}_A$. We also extend the definition of EA to SU(2) cases for the first time, where a carefully designed unitary transformation is required for ρ_A to properly address noncommuting conserved charges [6]. It is worth noting that $\Delta S_A \geq 0$ by definition and it only vanishes when ρ_A is block diagonal in the eigenbasis of the symmetry operator. Symmetry restoration as indicated by $\Delta S_A = 0$ is also a necessary condition for quantum thermalization due to the thermal equilibrium form of the mixed state. Because of the randomness in circuit configurations, we focus on the average EA, $\mathbb{E}[\Delta S_A]$. In the theoretical analysis, we utilize Rényi-2 EA, $\mathbb{E}[\Delta S_A^{(2)}]$, by replacing von Neumann entropy with Rényi-2 entropy for simplicity, which shows qualitatively the same behaviors as EA.

Based on the rigorous theoretical analysis and extensive numerical simulations, we have revealed that the subsystem symmetry restoration of the U(1)-symmetric circuits depends on initial states, which significantly differs from the case of random Haar circuits where the dynamics are agnostic of different initial states. Specifically, when starting from a tilted ferromagnetic state with a sufficiently small tilt angle θ, we demonstrate that the late-time EA remains nonzero in finite-size systems, indicating that the final state remains symmetry broken. The persistent symmetry breaking in finite-size systems is a universal feature for U(1) symmetric dynamics [6]. Conversely, the symmetry can always be restored when the initial state is more U(1)-asymmetric with a large tilt angle as long as $|A|/N < 1/2$.

More importantly, we have also observed the emergence of QME in the EA dynamics of U(1)-symmetric circuits, which is absent in random Haar circuits without symmetry. The emergence of QME in chaotic systems can be understood through the lens of quantum thermalization. The thermalization speed varies significantly across different charge sectors [6] (see also results for thermalization in U(1)-symmetric circuits [82,83]). Specifically, charge sectors with small Hilbert space dimensions do not obey ETH and thus do not thermalize or thermalize slowly. Consequently, symmetry restoration is slow when the subsystem of the initial state has a large overlap with the charge sector of a small dimension. Therefore, a QME occurs when the state with a smaller initial EA simultaneously has a larger overlap with the small charge sector resulting in slow thermalization, as in the case of the tilted ferromagnetic state. In sum, the mechanism behind QME is attributed to slower thermalization induced by more symmetric initial states. We have validated this theoretical understanding with different initial states and internal symmetries.

Setup—For the U(1) symmetry restoration, inspired by the previous studies [43], we adopt a tilted ferromagnetic state as the initial state (see more numerical results with different initial states in the SM [6]) defined as

$$|\psi_0(\theta)\rangle = e^{-i\frac{\theta}{2}\sum_j \sigma_j^y}|000\ldots0\rangle, \tag{2}$$

where σ_j^y is the Pauli-Y operator on jth qubit and the tilt angle θ determines the charge asymmetric level of the initial states: when $\theta = 0$, $|\psi_0(0)\rangle = |000\ldots0\rangle$ is U(1)-symmetric and $\Delta S_A = 0$ for any subsystem A. As θ increases, ΔS_A also increases until it reaches its maximal value at $\theta = \pi/2$.

As shown in Fig. 1, the initial state undergoes the unitary evolution of random quantum circuits with periodic

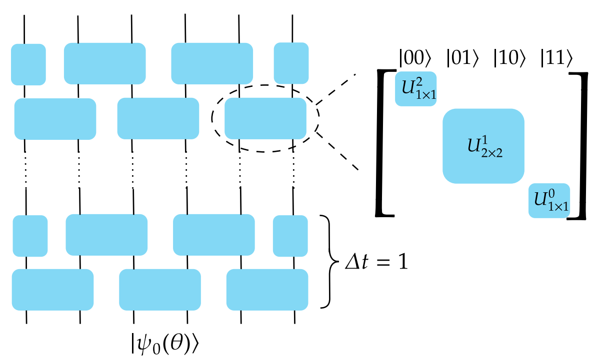

FIG. 1. Random circuit with six qubits. The initial state is chosen as the tilted ferromagnetic state and the blue rectangle represents random two-qubit gates in the even-odd brick-wall pattern. For U(1)-symmetric circuits, each two-qubit gate respects U(1) symmetry, resulting in a block diagonal structure for the unitary matrix of quantum gates.

boundary conditions where two-qubit gates are arranged in a brick-wall structure. In the case of nonsymmetric circuits, each two-qubit gate is randomly chosen from the Haar measure. For the U(1)-symmetric case, the matrix for each two-qubit gate is block diagonal as shown in Fig. 1 and each block is randomly sampled from the Haar measure. One discrete time step $\Delta t = 1$ includes two layers of two-qubit gates. We calculate the EA dynamics $\mathbb{E}[\Delta S_A]$ of subsystem A averaged over different circuit configurations to monitor the dynamical and steady behaviors of symmetry restoration.

Symmetry restoration in the long-time limit—We approximate the long-time limit of random circuit ensemble with a simpler ensemble $\mathbb{U}$ for a single random unitary U acting on all qubits to compute Rényi-2 EA [6,84–87]. For the nonsymmetric random circuit evolution, $\mathbb{U}$ constitutes a global 2-design for the Haar measure. Consequently, the average Rényi-2 EA at late time is [56]

$$\mathbb{E}[\Delta S_A^{(2)}] \approx -\log\left[\frac{1+2^{2|A|-N}/\sqrt{\pi|A|}}{1+2^{2|A|-N}}\right]. \quad (3)$$

With large N, $\mathbb{E}[\Delta S_A^{(2)}]$ approaches zero if $|A| < N/2$, while it sharply changes to a nonzero value $\log\sqrt{\pi|A|}$ if $|A| > N/2$. Therefore, the broken symmetry of subsystem A with $|A| < N/2$ can always be restored by the nonsymmetric random circuits at a late time, regardless of the initial states.

For the U(1)-symmetric random circuit evolution, $\mathbb{U}$ is a global 2-design for the composition of the Haar measures over each charge sector [85–87]. The average Rényi-2 EA can be accurately obtained by calculating certain summations of polynomial numbers of binomial coefficient products, which arise from counting charge numbers [6]. Under the condition of large system size and large tilt angle, a simplified analytical form of $\mathbb{E}[\Delta S_A^{(2)}]$ can be obtained by approximating the binomial coefficients with the Gaussian distributions,

$$\mathbb{E}[\Delta S_A^{(2)}] \approx -\log\left[\frac{1+g(\theta)^{2|A|-N}/\sqrt{\pi|A|}}{1+g(\theta)^{2|A|-N}}\right], \quad (4)$$

which resembles the nonsymmetric case in Eq. (3) except for the θ-dependent base factor

$$g(\theta) = 2\exp\left[-\frac{1}{2}\log^2\left(\tan^2\frac{\theta}{2}\right)\right]. \quad (5)$$

If $\theta = 0.5\pi$, Eq. (4) coincides with Eq. (3), and thus the late-time EA are the same as shown in Fig. 2(a). If the tilt angle remains large but deviates from 0.5π, Eq. (4) indicates that the main characteristic remains unchanged compared with the nonsymmetric case: the symmetry is restored for a small subsystem of $|A| < N/2$ but still broken for $|A| > N/2$.

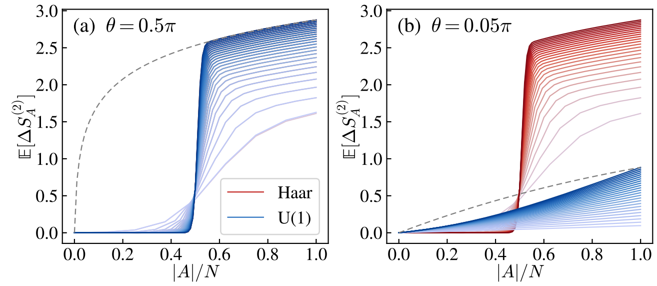

FIG. 2. The average Rényi-2 EA $\mathbb{E}[\Delta S_A^{(2)}]$ in the long-time limit of random circuit evolution starting from the tilted ferromagnetic initial states with tilt angle (a) $\theta = 0.5\pi$ and (b) $\theta = 0.05\pi$ versus the subsystem size $|A|$. The increasing intensity of colors represents increasing system sizes $N \in \{8, 12, \ldots, 100\}$. The blue and red lines represent the results for U(1)-symmetric random circuits and random circuits without symmetry restriction, respectively. The gray dashed lines represent the Rényi-2 EA for the initial states with $N = 100$.

However, if the tilt angle is sufficiently small, the Gaussian approximation fails. We rely on direct estimation of the summations of binomial coefficients [6], which shows that for small tilt angles such as $\theta < 0.1\pi$, $\mathbb{E}[\Delta S_A^{(2)}]$ will converge to a significant finite value in the long-time limit even for $|A| < N/2$ and large finite N as shown in Fig. 2(b). In other words, when the symmetry breaking in the tilted ferromagnetic initial state is relatively weak, it becomes challenging to fully restore the subsystem symmetry through the U(1)-symmetric random circuit evolution. Conversely, those initial states exhibiting more severe symmetry breaking can restore the symmetry successfully instead. This phenomenon is reminiscent of an extreme limit of the QME, where instead of restoring slowly, the symmetry does not fully restore for initial states with weak symmetry breaking. We remark that the persistent symmetry breaking is a universal feature of U(1)-restoring dynamics quenching a sufficiently weak symmetry-breaking tilted ferromagnetic state [6]. As shown in Fig. 3, there exists a critical value θ_c where on the small-θ side the symmetry is not restored, leaving a persistent symmetry-breaking behavior. It is worth noting that the above discussions only apply to the finite-size system as the critical value θ_c slowly varies with the system size N with a scaling of $\theta_c \approx 1.13\pi/\sqrt{N}$.

Quantum Mpemba effect in early-time dynamics—Now we proceed to consider the EA dynamics for different initial states with varying tilt angles θ. The numerical simulations are performed using the TensorCircuit package [88]. We observe that EA decays more rapidly as the tilt angle increases for U(1)-symmetric random quantum circuits, as illustrated in Fig. 4(a). Namely, a QME emerges in the symmetry restoration process. It is important to highlight that the presence of QME depends on the specific initial

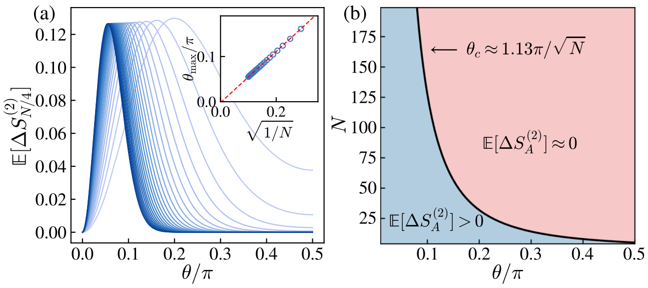

FIG. 3. (a) The average Rényi-2 EA $\mathbb{E}[\Delta S_A^{(2)}]$ in the long-time limit of U(1)-symmetric random circuit evolution at $|A| = N/4$ versus tilt angle θ. The increasing intensity of colors corresponds to $N \in \{8, 12, \ldots, 100\}$. The inset depicts the peak position $\theta_{\max}$ versus N. (b) The crossover for the symmetric restoration at $|A| < N/2$. The red and blue areas represent the symmetry-restored and persistent symmetry-breaking behaviors, respectively. The "critical value" $\theta_c \approx 2\theta_{\max}$ for the finite size crossover depends on N with a $1/\sqrt{N}$ scaling.

states and we observe the absence of QME with initial tilted Néel states as shown in Fig. 5(b). Nevertheless, we emphasize that the presence of QMEs is not a fine-tuned phenomenon. We further investigate the EA dynamics and observe QME with various sets of initial states, including tilted ferromagnetic states where the tilt angle on each qubit is randomly sampled from $[-W, W]$, which is more compatible with the experimental demonstration of QMEs on quantum devices since it does not require high precision state preparation [6].

On the contrary, for nonsymmetric random circuits, EA dynamics with different initial states coincide as depicted in Fig. 4(b). Consequently, although the EA still tends to zero at late times, the QME disappears trivially. This behavior can be understood in terms of the effective statistical model, where the initial state dependence has been eliminated as the inner product between different initial product states and the first layer of random unitary gates is constant [6].

In addition, we have conducted investigations on setups that incorporate additional symmetries, such as spatial or

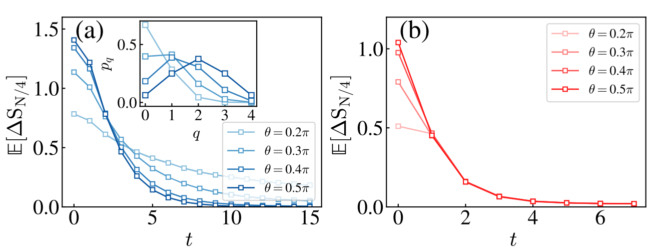

FIG. 4. (a),(b) show the EA dynamics of subsystem $A = [0, N/4]$ for random quantum circuits with and without U(1) symmetry respectively. We use $N = 16$ in the former case and $N = 8$ in the latter case. The inset of (a) shows the overlaps of ρ_A with different charge sectors ranging from $\{0, \ldots, N/4\}$. QME exists for U(1)-symmetric random circuits while it is absent in random circuits without any symmetry.

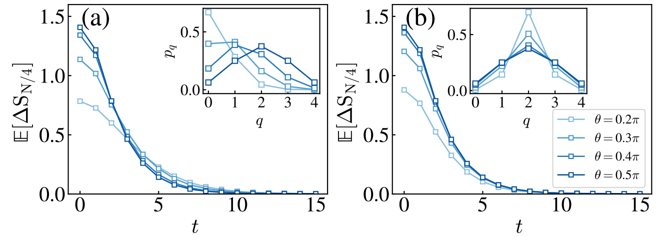

FIG. 5. EA dynamics with initial (a) tilted ferromagnetic state with a middle domain wall and (b) tilted Néel state. Insets show the overlaps of ρ_A with different charge sectors $q \in \{0, \ldots, N/4\}$. In the former case, the QME is present, whereas it is absent in the latter case, although the late-time behaviors are the same for both initial states based on the analytical results.

temporal translational symmetry (random Floquet circuit) [89–92]. We have found that thermalization accompanied by symmetry restoration and the QME persists in these setups as well [6]. Interestingly, the temporal translation symmetry slows down the symmetry restoration, consistent with the slow thermalization results in [91].

To understand the unified mechanism behind QME in generic chaotic systems, we first consider the overlaps between reduced density matrix ρ_A and different charge sectors, defined as $p_q = \mathrm{tr}(\Pi_q \rho_A \Pi_q)$. As shown in the inset of Fig. 4(a), the charge distribution is more peaked to the charge sector of a small dimension as the decreases of tilt angle θ, i.e., the initial state is more symmetric. Furthermore, the thermalization speeds for different charge sectors are conjectured to be different where charge sectors with $O(1)$ dimension generally fail the ETH and thermalize slowly. The conjecture of thermalization speed dependence on the Hilbert subspace dimension is numerically validated in the SM [6], which is of stand-alone importance toward a better understanding of quantum thermalization. Therefore, for tilted ferromagnetic states, weaker initial subsystem symmetry breaking is linked with slower thermalization speed via the larger overlap with small charge sectors. Consequently, QME occurs as the symmetry restoration is slower for more symmetric initial states.

This unified mechanism provides insights to identify the suitable initial states exhibiting QME beyond the tilted ferromagnetic state extensively investigated before. To further validate the thermalization explanation of QME, we investigate the EA dynamics of two different initial states: one is the tilted ferromagnetic state with a middle domain wall and the other is the tilted Néel state [6]. The steady-state EA should be the same for these two initial states as discussed in the SM [6]. However, QME is captured by the early-time behaviors of EA dynamics, making the local spin configurations of the initial states crucial due to the casual cone structure. The numerical results of overlaps between the two types of initial states and different charge sectors are shown in Fig. 5. For the tilted ferromagnetic state with a middle domain wall, the overlap distribution is similar to that of the tilted ferromagnetic state. Therefore, more symmetric states thermalize slower and the QME occurs. On the contrary, for the tilted Néel state, its charge distribution is always peaked to the largest charge sector and thus it strongly obeys the ETH regardless of tilt angles θ. Consequently, the thermalization speed is essentially unchanged and the QME is absent.

Moreover, we also investigate the symmetry restoration dynamics in quantum circuits with SU(2) and Z_2 symmetries. We extend the definition of EA for SU(2) symmetry and identify the presence (absence) of QME in the SU(2) (Z_2) symmetric circuits [6]. The unified mechanism above also provides insights into these different internal symmetry cases. For Z_2 symmetry, there are only two equally large Hilbert subspaces, and the thermalization speeds of different initial states are expected to be similar with no QME. In the SU(2)-symmetry case, the dimensions of different symmetry sectors vary from constant to exponential scaling like the U(1) case. Therefore, QME can be observed for carefully designed initial states satisfying the criteria above.

Conclusions and discussions—In this Letter, we have presented a rigorous and comprehensive theoretical analysis of subsystem symmetry restoration under the evolution of random quantum circuits respecting the U(1) symmetry. Our findings reveal that U(1)-symmetric circuits hinder the U(1) symmetry restoration when the input is a tilted ferromagnetic initial state with a small tilt angle θ. Conversely, the symmetry can always be restored when the tilt angle is large, i.e., the initial state is more U(1)-asymmetric. These results highlight the distinctions between U(1)-symmetric and nonsymmetric circuits in terms of symmetry restoration and quantum thermalization.

More importantly, besides the late-time analytical results, we have numerically investigated the early-time dynamics of symmetry restoration and provided a unified understanding of QME in generic chaotic systems in the context of quantum thermalization. We have validated this theoretical understanding via the correct predictions of the presence and absence of QME for various initial states and internal symmetries that have not been explored before.

There are various interesting questions worth further investigation—for example, the comprehensive extension of the symmetry restoration of other internal symmetries in both integrable and chaotic systems. Additionally, the investigation of symmetry restoration and the QME in the many-body localized systems is lacking. Addressing this gap will significantly enhance our theoretical comprehension of the mechanisms underlying the QME across diverse systems [93].

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